

Jet Riemann-Lagrange Geometry Applied to Evolution DEs Systems from Economy

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Abstract

The aim of this paper is to construct a natural Riemann-Lagrange differential geometry on 1-jet spaces, in the sense of nonlinear connections, generalized Cartan connections, d-torsions, d-curvatures, jet electromagnetic fields and jet Yang-Mills energies, starting from some given non-linear evolution DEs systems modelling economic phenomena, like the Kaldor model of the bussines cycle or the Tobin-Benhabib-Miyao model regarding the role of money on economic growth.

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1 Historical aspects

According to Olver and Udrishte opinions expressed in [9], [13] and in private discussions, a lot of applicative problems coming from Physics [13], Biology [6] or Economics [14] can be modelled on 1-jet spaces $J^1(T, M)$, where T^p is a smooth "multi-time" manifold of dimension p and M^n is a smooth "spatial" manifold of dimension n . In a such context, a lot of authors (for example, Asanov [1], Saunders [11], Vondra [15] and many others) studied the *contravariant differential geometry* of 1-jet spaces. Proceeding with the geometrical studies of Asanov, the author of this paper has recently developed the *Riemann-Lagrange geometry of 1-jet spaces* [5], which is a natural extension on 1-jet spaces of the well known *Lagrange geometry of the tangent bundle* due to Miron and Anastasiei [4].

It is important to note that the Riemann-Lagrange geometry of 1-jet spaces contains many fruitful ideas for the geometrical interpretation of the solutions of a given DEs or PDEs system [7]. For instance, Udrishte proved in [13] that the orbits of a given vector field may be regarded as horizontal geodesics in a suitable Riemann-Lagrange geometrical structure, solving in this way an old open problem suggested by Poincaré [10]: *Find the geometric structure which transforms the field lines of a given vector field into geodesics*.

In the sequel, we present the main geometrical ideas used by Udrishte in order to solve the open problem of Poincaré. For more details, the reader is invited to consult the works [13] and [14].

In this direction, let us consider a Riemannian manifold $(M^n, \varphi_{ij}(x))$ and let us fix an arbitrary vector field $X = (X^i(x))$ on M . Obviously, the vector field X produces the first order DEs system (*dynamical system*)

$$\frac{dx^i}{dt} = X^i(x(t)), \quad \forall i = \overline{1, n}. \quad (1.1)$$

Differentiating the first order DEs system (1.1) and making a convenient arranging of the terms involved, via the vector field X , the Riemannian metric φ_{ij} and its Christoffel symbols γ_{jk}^i , Udriște constructs a second order prolongation (*single-time geometric dynamical system*) having the form

$$\frac{d^2 x^i}{dt^2} + \gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = F_j^i \frac{dx^j}{dt} + \varphi^{ih} \varphi_{kj} X^j \nabla_h X^k, \quad \forall i = \overline{1, n}, \quad (1.2)$$

where ∇ is the Levy-Civita connection of the Riemannian manifold (M, φ) and

$$F_j^i = \nabla_j X^i - \varphi^{ih} \varphi_{kj} \nabla_h X^k.$$

Remark 1.1 *Note that any solution of class C^2 of the first order DEs system (1.1) is also a solution for the second order DEs system (1.2). Conversely, this statement is not true.*

The second order DEs system (1.2) is important because it is equivalent with the Euler-Lagrange equations of that so-called the *least squares Lagrangian function*

$$LS : TM \rightarrow \mathbb{R}_+,$$

given by

$$LS(x, y) = \frac{1}{2} \varphi_{ij}(x) [y^i - X^i(x)] [y^j - X^j(x)]. \quad (1.3)$$

It is obvious now that the field lines of class C^2 of the vector field X are the *global minimum points* of the *least squares energy action* attached to LS , so these field lines are solutions of the Euler-Lagrange equations produced by LS . Because the Euler-Lagrange equations of LS are exactly the equations (1.2), Udriște asserts that the solutions of class C^2 of the first order DEs system (1.1) are *horizontal geodesics* on the *Riemann-Lagrange manifold*

$$(\mathbb{R} \times M, 1 + \varphi, N^i_j) = \gamma_{jk}^i y^k - F_j^i.$$

Remark 1.2 *The author of this paper believe that the preceding least squares variational method for the geometrical study of the DEs system (1.1) can be reduced to a natural extension of the following well known and simple idea coming from linear algebra: **In any Euclidian vector space (V, \langle, \rangle) the following equivalence holds good:***

$$v = 0_V \Leftrightarrow \|v\| = 0.$$

Using as a pattern the geometrical Udriște's ideas, in what follows we expose the main geometrical results on 1-jet spaces that, in our opinion, characterize a given first order non-linear DEs system regarded as an ordinary differential system on an 1-jet space $J^1(T, M)$, where $T \subset \mathbb{R}$.

2 Jet Riemann-Lagrange geometry produced by a non-linear DEs system of order one

Let $T = [a, b] \subset \mathbb{R}$ be a compact interval of the set of real numbers and let us consider the jet fibre bundle of order one

$$J^1(T, \mathbb{R}^n) \rightarrow T \times \mathbb{R}^n, \quad n \geq 2,$$

whose local coordinates (t, x^i, x_1^i) , $i = \overline{1, n}$, transform by the rules

$$\tilde{t} = \tilde{t}(t), \quad \tilde{x}^i = \tilde{x}^i(x^j), \quad \tilde{x}_1^i = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{d\tilde{t}} \cdot x_1^j.$$

Remark 2.1 *From a physical point of view, the coordinate t has the physical meaning of **relativistic time**, the coordinates $(x^i)_{i=\overline{1, n}}$ represent **spatial coordinates** and the coordinates $(x_1^i)_{i=\overline{1, n}}$ have the physical meaning of **relativistic velocities**.*

Let us consider that $X = \left(X_{(1)}^{(i)}(x^k) \right)$ is an arbitrary d-tensor field on the 1-jet space $J^1(T, \mathbb{R}^n)$, whose local components transform by the rules

$$\tilde{X}_{(1)}^{(i)} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{d\tilde{t}} \cdot X_{(1)}^{(j)}.$$

Obviously, the d-tensor field X produces the jet DEs system of order one (*jet dynamical system*)

$$x_1^i = X_{(1)}^{(i)}(x^k(t)), \quad \forall i = \overline{1, n}, \quad (2.1)$$

where $c(t) = (x^i(t))$ is an unknown curve on \mathbb{R}^n (i. e., a jet field line of the d-tensor field X) and we used the notation

$$x_1^i \stackrel{\text{not}}{=} \dot{x}^i = \frac{dx^i}{dt}, \quad \forall i = \overline{1, n}.$$

Supposing now that we have the Euclidian structures $(T, 1)$ and $(\mathbb{R}^n, \delta_{ij})$, where δ_{ij} are the Kronecker symbols, then the jet first order DEs system (2.1) automatically produces the *jet least squares Lagrangian function*

$$JLS : J^1(T, \mathbb{R}^n) \rightarrow \mathbb{R}_+,$$

expressed by

$$JLS(x^k, x_1^k) = \sum_{i=1}^n \left[x_1^i - X_{(1)}^{(i)}(x) \right]^2, \quad (2.2)$$

where $x = (x^k)_{k=\overline{1, n}}$. Of course, the *global minimum points* of the *jet least squares energy action*

$$\mathbb{E}(c(t)) = \int_a^b JLS(x^k(t), \dot{x}^k(t)) dt$$

are exactly the solutions of class C^2 of the jet first order DEs system (2.1). In other words, the solutions of class C^2 of the jet DEs system of order one (2.1) verify the second order Euler-Lagrange equations produced by *JLS* (*jet geometric dynamics*).

Remark 2.2 *Because a Riemann-Lagrange geometry on $J^1(T, \mathbb{R}^n)$ produced by the jet least squares Lagrangian function *JLS*, via its second order Euler-Lagrange equations, in the sense of non-linear connection, generalized Cartan connection, d-torsions, d-curvatures, jet electromagnetic field and jet Yang-Mills energy, is now completely done in the papers [5], [6] and [7], it follows that we may regard *JLS* as a natural geometrical substitut on $J^1(T, \mathbb{R}^n)$ for the jet first order DEs system (2.1).*

In this context, we introduce the following notion:

Definition 2.3 *Any geometrical object on $J^1(T, \mathbb{R}^n)$, which is produced by the jet least squares Lagrangian function *JLS*, via its second order Euler-Lagrange equations, is called **geometrical object produced by the jet first order DEs system (2.1)**.*

In order to expose the main jet Riemann-Lagrange geometrical objects that characterize the jet first order DEs system (2.1), we use the following matriceal Jacobian notation:

$$J(X_{(1)}) = \left(\frac{\partial X_{(1)}^{(i)}}{\partial x^j} \right)_{i,j=\overline{1,n}}.$$

In this context, the following geometrical result, which is proved in [6] and, for more general cases, in [5] and [7], holds good.

Theorem 2.4 *(i) The **canonical non-linear connection on $J^1(T, \mathbb{R}^n)$ produced by the jet first order DEs system (2.1)** has the local components*

$$\Gamma = \left(0, N_{(1)j}^{(i)} \right),$$

where $N_{(1)j}^{(i)}$ are the entries of the matrix

$$N_{(1)} = \left(N_{(1)j}^{(i)} \right)_{i,j=\overline{1,n}} = -\frac{1}{2} [J(X_{(1)}) - {}^T J(X_{(1)})].$$

*(ii) All adapted components of the **canonical generalized Cartan connection CT produced by the jet first order DEs system (2.1)** vanish.*

*(iii) The effective adapted components $R_{(1)jk}^{(i)}$ of the **torsion d-tensor T** of the canonical generalized Cartan connection CT **produced by the jet first order DEs system (2.1)** are the entries of the matrices*

$$R_{(1)k} = \frac{\partial}{\partial x^k} [N_{(1)}], \quad \forall k = \overline{1,n},$$

where

$$R_{(1)k} = \left(R_{(1)jk}^{(i)} \right)_{i,j=\overline{1,n}}, \quad \forall k = \overline{1,n}.$$

(iv) All adapted components of the **curvature** d-tensor \mathbf{R} of the canonical generalized Cartan connection CT **produced by the jet first order DEs system (2.1)** vanish.

(v) The **geometric electromagnetic distinguished 2-form produced by the jet first order DEs system (2.1)** has the expression

$$F = F_{(i)j}^{(1)} \delta x_1^i \wedge dx^j,$$

where

$$\delta x_1^i = dx_1^i + N_{(1)k}^{(i)} dx^k, \quad \forall i = \overline{1,n},$$

and the adapted components $F_{(i)j}^{(1)}$ are the entries of the matrix

$$F^{(1)} = \left(F_{(i)j}^{(1)} \right)_{i,j=\overline{1,n}} = -N_{(1)}.$$

(vi) The adapted components $F_{(i)j}^{(1)}$ of the geometric electromagnetic d-form F produced by the jet first order DEs system (2.1) verify the **generalized Maxwell equations**

$$\sum_{\{i,j,k\}} F_{(i)j||k}^{(1)} = 0,$$

where $\sum_{\{i,j,k\}}$ represents a cyclic sum and

$$F_{(i)j||k}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial x^k}$$

means the horizontal local covariant derivative produced by the Berwald connection $B\Gamma_0$ on $J^1(T, \mathbb{R}^n)$. For more details, please consult [5].

(vii) The **geometric jet Yang-Mills energy produced by the jet first order DEs system (2.1)** is given by the formula

$$EYM(x) = \frac{1}{2} \cdot \text{Trace} \left[F^{(1)} \cdot {}^T F^{(1)} \right],$$

where

$$F^{(1)} = \left(F_{(i)j}^{(1)} \right)_{i,j=\overline{1,n}}.$$

In the next Sections, we apply the above jet Riemann-Lagrange geometrical results to certain evolution equations that govern diverse phenomena in Economy, extending in this way the geometrical studies initiated by Udriște, Ferrara and Opreș in the book [14].

3 Jet Riemann-Lagrange geometry for Kaldor non-linear cyclical model in business

The *national revenue* $Y(t)$ and the *capital stock* $K(t)$, where $t \in [a, b]$, are the state variables of the Kaldor non-linear model of the business cycle. The kinetic Kaldor model of a commercial cycle belongs to the category of business cycles described by the *Kaldor flow* (for more details, please see [3], [14])

$$\begin{cases} \frac{dY}{dt} = s[I(Y, K) - S(Y, K)] \\ \frac{dK}{dt} = I(Y, K) - qK, \end{cases} \quad (3.1)$$

where

- $I = I(Y, K)$ is a given differentiable *investment function*, which verifies some *economic-mathematical Kaldor conditions*;
- $S = S(Y, K)$ is a given differentiable *saving function*, which verifies some *economic-mathematical Kaldor conditions*;
- $s > 0$ is an adjustment constant parameter which measures the reaction of the model with respect to the difference between the investment function and the saving function;
- $q \in (0, 1)$ is a constant representing the depreciation coefficient of capital.

Remark 3.1 *Details upon the **Kaldor economic-mathematical conditions** imposed to the given functions I and S find in [3] and [14]. From the point of view of the jet Riemann-Lagrange geometry produced by the Kaldor evolution equations, geometry that we will describe in the sequel, the economic-mathematical hypotheses of Kaldor can be neglected because all our geometrical informations are concentrated in the Kaldor flow (3.1).*

The Riemann-Lagrange geometrical behavior on the 1-jet space $J^1(T, \mathbb{R}^2)$ of the Kaldor economic evolution model is described in the following result:

Theorem 3.2 *(i) The **canonical non-linear connection on $J^1(T, \mathbb{R}^2)$** produced by the **Kaldor flow (3.1)** has the local components*

$$\hat{\Gamma} = \left(0, \hat{N}_{(1)j}^{(i)} \right),$$

where, if I_Y , I_K and S_K are the partial derivatives of the functions I and S , then $\hat{N}_{(1)j}^{(i)}$ are the entries of the matrix

$$\hat{N}_{(1)} = \begin{pmatrix} 0 & \frac{1}{2}[I_Y - s(I_K - S_K)] \\ -\frac{1}{2}[I_Y - s(I_K - S_K)] & 0 \end{pmatrix}.$$

(ii) All adapted components of the **canonical generalized Cartan connection** $C\hat{\Gamma}$ **produced by the Kaldor flow (3.1)** vanish.

(iii) All adapted components of the **torsion** d -tensor \hat{T} of the canonical generalized Cartan connection $C\hat{\Gamma}$ **produced by the Kaldor flow (3.1)** are zero, except

$$\begin{aligned}\hat{R}_{(1)21}^{(1)} &= -\hat{R}_{(1)11}^{(2)} = \frac{1}{2}[I_{YY} - s(I_{YK} - S_{YK})], \\ \hat{R}_{(1)22}^{(1)} &= -\hat{R}_{(1)12}^{(2)} = \frac{1}{2}[I_{YK} - s(I_{KK} - S_{KK})],\end{aligned}$$

where I_{YY} , I_{YK} , I_{KK} , S_{YK} and S_{KK} are the second partial derivatives of the functions I and S .

(iv) All adapted components of the **curvature** d -tensor \hat{R} of the canonical generalized Cartan connection $C\hat{\Gamma}$ **produced by the Kaldor flow (3.1)** vanish.

(v) The **geometric electromagnetic distinguished 2-form produced by the Kaldor flow (3.1)** has the expression

$$\hat{F} = \hat{F}_{(i)j}^{(1)} \delta x_1^i \wedge dx^j,$$

where

$$\delta x_1^i = dx_1^i + \hat{N}_{(1)k}^{(i)} dx^k, \quad \forall i = \overline{1, 2},$$

and the adapted components $\hat{F}_{(i)j}^{(1)}$ are the entries of the matrix

$$\hat{F}^{(1)} = -\hat{N}_{(1)} = \begin{pmatrix} 0 & -\frac{1}{2}[I_Y - s(I_K - S_K)] \\ \frac{1}{2}[I_Y - s(I_K - S_K)] & 0 \end{pmatrix}.$$

(vi) The **economic geometric Yang-Mills energy produced by the Kaldor flow (3.1)** is given by the formula

$$EYM^{Kaldor}(Y, K) = \frac{1}{4} [I_Y - s(I_K - S_K)]^2.$$

Proof. Let us regard the Kaldor flow (3.1) as a particular case of the jet first order DEs system (2.1) on the 1-jet space $J^1(T, \mathbb{R}^2)$, taking

$$n = 2, \quad x^1 = Y, \quad x^2 = K$$

and putting

$$X_{(1)}^{(1)}(x^1, x^2) = s[I(x^1, x^2) - S(x^1, x^2)] \quad \text{and} \quad X_{(1)}^{(2)}(x^1, x^2) = I(x^1, x^2) - qx^2.$$

Now, taking into account that we have the Jacobian matrix

$$\begin{aligned}J(X_{(1)}) &= \begin{pmatrix} s[I_{x^1} - S_{x^1}] & s[I_{x^2} - S_{x^2}] \\ I_{x^1} & I_{x^2} - q \end{pmatrix} \\ &= \begin{pmatrix} s[I_Y - S_Y] & s[I_K - S_K] \\ I_Y & I_K - q \end{pmatrix},\end{aligned}$$

and using the Theorem 2.4, we obtain what we were looking for. ■

Remark 3.3 (Open problem) *The Yang-Mills economic energetical curves of constant level produced by the Kaldor flow (3.1), which are different by the empty set, are the curves in the plane YOK having the implicit equations*

$$\mathcal{C}_C : [I_Y - s(I_K - S_K)]^2 = 4C,$$

where $C \geq 0$. Is it possible as the shapes of the plane curves \mathcal{C}_C to offer economic interpretations for economists?

4 Jet Riemann-Lagrange geometry for Tobin-Benhabib-Miyao economic evolution model

The Tobin mathematical model [12] regarding the role of money on economic growth was extended by Benhabib and Miyao [2] by incorporating the role of some expectation constant parameters. Thus, the Tobin-Benhabib-Miyao (TBM) economic model relies on the variables $k(t)$ = the capital labor ratio, $m(t)$ = the money stock per head, $q(t)$ = the expected rate of inflation, whose evolution in time is given by the TBM flow [14]

$$\begin{cases} \frac{dk}{dt} = sf(k(t)) - (1-s)[\theta - q(t)]m(t) - nk(t) \\ \frac{dm}{dt} = m(t) \{ \theta - n - q(t) - \varepsilon[m(t) - l(k(t), q(t))] \} \\ \frac{dq}{dt} = \mu\varepsilon[m(t) - l(k(t), q(t))], \end{cases} \quad (4.1)$$

where the $f(k)$ and $l(k, q)$ are some given differentiable real functions and $s, \theta, n, \mu, \varepsilon$ are expectation parameters: s = saving ratio, θ = rate of money expansion, n = population growth rate, μ = speed of adjustement of expectations, ε = speed of adjustement of price level.

Remark 4.1 *From the point of view of economists, the actual rate of inflation in the TBM economic model is given by the formula [14]*

$$\bar{p}(t) = \varepsilon[m(t) - l(k(t), q(t))] + q(t).$$

In what follows, we apply our jet Riemann-Lagrange geometrical results to the TBM flow (4.1). In this context, we obtain:

Theorem 4.2 (i) *The canonical non-linear connection on $J^1(T, \mathbb{R}^3)$ produced by the TBM flow (4.1) has the local components*

$$\tilde{\Gamma} = \left(0, \tilde{N}_{(1)j}^{(i)} \right),$$

where, if l_k and l_q are the partial derivatives of the function l , then $\check{N}_{(1)j}^{(i)}$ are the entries of the matrix

$$\check{N}_{(1)} = -\frac{1}{2} \begin{pmatrix} 0 & -(1-s)(\theta-q) - \varepsilon m l_k & (1-s)m + \mu \varepsilon l_k \\ (1-s)(\theta-q) + \varepsilon m l_k & 0 & -m + \varepsilon m l_q - \mu \varepsilon \\ -(1-s)m - \mu \varepsilon l_k & m - \varepsilon m l_q + \mu \varepsilon & 0 \end{pmatrix}.$$

(ii) All adapted components of the **canonical generalized Cartan connection $C\check{\Gamma}$ produced by the TBM flow (4.1)** vanish.

(iii) The effective adapted components of the **torsion d -tensor \check{T} of the canonical generalized Cartan connection $C\check{\Gamma}$ produced by the TBM flow (4.1)** are the entries of the matrices

$$\check{R}_{(1)1} = -\frac{1}{2} \begin{pmatrix} 0 & -\varepsilon m l_{kk} & \mu \varepsilon l_{kk} \\ \varepsilon m l_{kk} & 0 & \varepsilon m l_{kq} \\ -\mu \varepsilon l_{kk} & -\varepsilon m l_{kq} & 0 \end{pmatrix},$$

$$\check{R}_{(1)2} = -\frac{1}{2} \begin{pmatrix} 0 & -\varepsilon l_k & 1-s \\ \varepsilon l_k & 0 & -1 + \varepsilon l_q \\ -1+s & 1-\varepsilon l_q & 0 \end{pmatrix}$$

and

$$\check{R}_{(1)3} = -\frac{1}{2} \begin{pmatrix} 0 & 1-s - \varepsilon m l_{kq} & \mu \varepsilon l_{kq} \\ -1+s + \varepsilon m l_{kq} & 0 & \varepsilon m l_{qq} \\ -\mu \varepsilon l_{kq} & -\varepsilon m l_{qq} & 0 \end{pmatrix},$$

where l_{kk} , l_{kq} and l_{qq} are the second partial derivatives of the function l and

$$\check{R}_{(1)k} = \left(\check{R}_{(1)jk}^{(i)} \right)_{i,j=\overline{1,3}}, \quad \forall k = \overline{1,3}.$$

(iv) All adapted components of the **curvature d -tensor \check{R} of the canonical generalized Cartan connection $C\check{\Gamma}$ produced by the TBM flow (4.1)** vanish.

(v) The **geometric electromagnetic distinguished 2-form produced by the TBM flow (4.1)** has the expression

$$\check{F} = \check{F}_{(i)j}^{(1)} \delta x_1^i \wedge dx^j,$$

where

$$\delta x_1^i = dx_1^i + \check{N}_{(1)k}^{(i)} dx^k, \quad \forall i = \overline{1,3},$$

and the adapted components $\check{F}_{(i)j}^{(1)}$ are the entries of the matrix

$$\check{F}^{(1)} = -\check{N}_{(1)}.$$

(vi) The *economic geometric Yang-Mills energy produced by the TBM flow (4.1)* is given by the formula

$$EYM^{TBM}(k, m, q) = \frac{1}{4} \left\{ [(1-s)(\theta - q) + \varepsilon m l_k]^2 + [(1-s)m + \mu \varepsilon l_k]^2 + [m - \varepsilon m l_q + \mu \varepsilon]^2 \right\}.$$

Proof. We regard the TBM flow (4.1) as a particular case of the jet first order DEs system (2.1) on the 1-jet space $J^1(T, \mathbb{R}^3)$, taking

$$n = 3, \quad x^1 = k, \quad x^2 = m, \quad x^3 = q$$

and putting

$$X_{(1)}^{(1)}(x^1, x^2, x^3) = s f(x^1) - (1-s)[\theta - x^3]x^2 - n x^1,$$

$$X_{(1)}^{(2)}(x^1, x^2, x^3) = x^2 \{ \theta - n - x^3 - \varepsilon [x^2 - l(x^1, x^3)] \}$$

and

$$X_{(1)}^{(3)}(x^1, x^2, x^3) = \mu \varepsilon [x^2 - l(x^1, x^3)].$$

It follows that we have the Jacobian matrix

$$\begin{aligned} J(X_{(1)}) &= \begin{pmatrix} s f'(x^1) - n & -(1-s)(\theta - x^3) & (1-s)x^2 \\ \varepsilon x^2 l_{x^1} & -2\varepsilon x^2 + \theta - x^3 - n + \varepsilon l(x^1, x^3) & -x^2 + \varepsilon x^2 l_{x^3} \\ -\mu \varepsilon l_{x^1} & \mu \varepsilon & -\mu \varepsilon l_{x^3} \end{pmatrix} \\ &= \begin{pmatrix} s f'(k) - n & -(1-s)(\theta - q) & (1-s)m \\ \varepsilon m l_k & -2\varepsilon m + \theta - q - n + \varepsilon l(k, q) & -m + \varepsilon m l_q \\ -\mu \varepsilon l_k & \mu \varepsilon & -\mu \varepsilon l_q \end{pmatrix}, \end{aligned}$$

where f' is the derivative of the function f . In conclusion, using the Theorem 2.4, we find the required result. ■

Remark 4.3 (Open problem) The *Yang-Mills economic energetical surfaces of constant level produced by the TBM flow (4.1)*, which are different by the empty set, have in the system of axis $Okmq$ the implicit equations

$$\Sigma_C : [(1-s)(\theta - q) + \varepsilon m l_k]^2 + [(1-s)m + \mu \varepsilon l_k]^2 + [m - \varepsilon m l_q + \mu \varepsilon]^2 = 4C,$$

where $C \geq 0$. Is it possible as the geometry of the surfaces Σ_C to contains valuable economic informations for economists?

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